Available online at www.sciencedirect.com
SCIENCE DIRECT®

ACADEMIC PRESS

Journal of Approximation Theory 120 (2003) 296–308

Journal of Approximation Theory

http://www.elsevier.com/locate/jat

The retraction constant in some Banach spaces

Marco Baronti,[a,*] Emanuele Casini,[b] and Carlo Franchetti[c]

[a] Dimet, Università degli Studi di Genova, Piazzale Kennedy, Genova 16129, Italy
[b] Dipartimento di Scienze Chimiche, Fisiche e Matematiche, Università degli Studi dell'Insubria, Via Valleggio 5, Como 22100, Italy
[c] Dipartimento di Matematica Applicata, Università degli Studi di Firenze, Via di S. Marta 3, Firenze 50139, Italy

Received 16 November 2001; accepted in revised form 18 October 2002

Abstract

We give new sharper estimations for the retraction constant in some Banach spaces.

Keywords: Retractions; Lipschitz maps; Minimal displacement

1. Introduction and notations

It is well known that Brouwer's fixed point theorem has the following equivalent form: If X is a finite-dimensional Banach space then there is no continuous retraction from the closed unit ball $B(X)$ onto the unit sphere $S(X)$ (R is a retraction if $Rx = x$ for all $x \in S(X)$). After some partial results (see [G-K] for an optimal list of references), Benyamini and Sternfeld solved completely the problem of existence of Lipschitzian retractions in the infinite-dimensional case by proving the following result:

Theorem 1 (Benyamini and Sternfeld [B-S]). *There exists an universal constant $\mathscr{K}$ such that for every infinite-dimensional Banach space X there exists a Lipschitzian retraction R from the unit ball $B(X)$ onto the unit sphere $S(X)$ with Lipschitz constant less than $\mathscr{K}$.*

*Corresponding author.
E-mail addresses: baronti@dimet.unige.it (M. Baronti), emanuele.casini@uninsubria.it (E. Casini), franchetti@dma.unifi.it (C. Franchetti).

0021-9045/02/$ - see front matter
doi:10.1016/S0021-9045(02)00037-0

This theorem suggests some quantitative problems. First of all it seems that no significant lower and no upper bound for the best constant $\mathscr{K}$ that appears in Theorem 1 is known. Given an infinite-dimensional Banach space X, the retraction constant $k_0(X)$ is defined as the infimum of such k for which there exists a retraction $R: B(X) \to S(X)$ in $\mathscr{L}(k)$, where $\mathscr{L}(k)$ is the class of Lipschitz maps with constant k. The only general result states that (see [G-K]): $k_0(X) \geqslant 3$ for every space X. Some estimations of $k_0(X)$ for some classical Banach space are available in the literature. Here is a list of known results:

$$
\begin{aligned}
&4 \leqslant k_0(l_1) < 31.64 && (\text{see [B1]}),\\
&k_0(c_0) < 35.18 && (\text{see [B3]}),\\
&k_0(L^1[0,1]) < 9.43 && (\text{see [G-K]}),\\
&k_0(C([a,b])) < 23.31 && (\text{see [G2]}),\\
&4.55 < k_0(H) < 64.25 && (\text{see [G-K, K-W]}).
\end{aligned}
$$

The aim of this paper is to improve the known upper bounds for the retraction constant $k_0(X)$ in some classical Banach spaces.

Let X be an infinite-dimensional Banach space X. For any $T: B(X) \to B(X)$ the minimal displacement η_T of T is defined by

$$\eta_T = \inf\{\|x - Tx\|: x \in B(X)\}.$$

The minimal displacement characteristic of X is

$$\psi_X(k) = \sup_{T \in \mathscr{L}(k)} \eta_T.$$

This function, for $k > 1$, satisfies

$$0 < \psi_X(k) \leqslant 1 - \frac{1}{k}.$$

The study of this function started in [G1] where the upper bound is proved while the positiveness of ψ_X is a consequence of the Benyamini–Sternfeld result. This and other properties of the minimal displacement characteristic can be found in the book of Kirk and Goebel [G-K] (see also [B1,B2]). We recall the definition of the radial projection P_R from a Banach space X onto its unit ball $B(X)$:

$$P_R(x) = \begin{cases} x & \text{if } \|x\| \leqslant 1,\\ x/\|x\| & \text{if } 1 < \|x\|. \end{cases}$$

P_R is a Lipschitz function on X. The best Lipschitz constant for P_R is denoted by $h(X)$ and is called the radial constant of the space X. It is easily seen that $1 \leqslant h(X) \leqslant 2$. It is also known that if $\dim X > 2$, X is a Hilbert space if and only if $h(X) = 1$. The exact values of $h(X)$ in L_p spaces are given in [F].

Note the following useful fact: for elements $x, y \in X$, if $\|x\|, \|y\| \geqslant r > 0$ we have (see for e.g. [B-G])

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \left\|P_R\left(\frac{x}{r}\right) - P_R\left(\frac{y}{r}\right)\right\| \leqslant \frac{h(X)}{r}\|x - y\|.$$

A subset W, with at least two points, of a normed space X is *δ-dispersed* if $\|x - y\| > \delta$ for each pair x, y of distinct points of W. A subset W of a normed space X is *proximinal* if for each $x \in X$ there exists an element $w(x) \in W$ such that $\|x - w(x)\| = \operatorname{dist}(x, W)$.

2. A minimum principle

In this section, we formulate a simple minimum principle which we will use later for the construction of retractions. The next proposition is probably known but we were not able to find references.

Define

$$G = \{g \in C^1[0,1]:\ g(0) = \gamma,\ \ g(1) = 1\},$$

$$\Phi : G \to C[0,1]; \quad \Phi(g) = (g-1)' - L(g-1),$$

where γ and $L > 0$ are fixed constants.

Proposition 1.

$$\inf\{\|\Phi(g)\|:\ g \in G\} = \frac{L|1-\gamma|}{1-e^{-L}} = \|\Phi(\bar{g})\|,$$

where the norm is the usual supremum norm and

$$\bar{g}(t) = 1 + e^{Lt}\left[\int_0^t e^{-Ls}\frac{L(1-\gamma)}{1-e^{-L}}\,ds + (\gamma - 1)\right] = 1 + \frac{(1-\gamma)(e^{L(t-1)} - 1)}{1-e^{-L}}.$$

Proof. Set $\Phi(g) = v$, then

$$g(t) - 1 = e^{Lt}\left[\int_0^t e^{-Ls}v(s)\,ds + (\gamma - 1)\right]$$

and since $g(1) = 1$ we get

$$\int_0^1 e^{-Ls}v(s)\,ds = 1 - \gamma.$$

If the function v has to be constant, say v^*, then

$$v^* = (1-\gamma)\left[\int_0^1 e^{-Ls}\,ds\right]^{-1} = \frac{L(1-\gamma)}{1-e^{-L}}.$$

We now have for any $g \in G$

$$|1-\gamma| \leqslant \int_0^1 e^{-Ls} ||v||\, ds = ||\Phi(g)|| \frac{1-e^{-L}}{L}$$

so that

$$||\Phi(g)|| \geqslant \frac{L|1-\gamma|}{1-e^{-L}} = ||\Phi(\bar{g})||. \qquad \square$$

3. Construction of a retraction

Now we describe a general method of construction of a retraction, the origin of which is in [B-G]. In Proposition 1 assume that $\gamma < 1$, and let $\bar{g}$ be the corresponding function obtained with the minimum principle.

Proposition 2. *Let $T : B(X) \to B(X)$ be a map in $\mathscr{L}(L)$. Suppose that there exists a function $d : [0,1] \to R$ such that*

$$||x - Tx|| \geqslant d(||x||)$$

for any $x \in B(X)$.

Define, for $x \in B(X)$

$$Ax = x - (1 - \bar{g}(||x||))\, Tx,$$

then (trivially) $Ax = x$ for $x \in S(X)$ and moreover

(i) *For any $x \in B(X)$*

$$||Ax|| \geqslant M,$$

where

$$M = \min_{0 \leqslant t \leqslant 1} \max[t - 1 + \bar{g}(t), d(t) - \bar{g}(t)].$$

(ii) *$A \in \mathscr{L}(C)$ where*

$$C = 1 + \frac{L(1-\gamma)}{1-e^{-L}}.$$

(iii) *If $M > 0$ then $Rx = \frac{Ax}{||Ax||}$ defines a lipschitz retraction of $B(X)$ onto $S(X)$ in $\mathscr{L}(Q)$, where*

$$Q = \frac{h(X)}{M}\left(1 + \frac{L(1-\gamma)}{1-e^{-L}}\right). \tag{1}$$

Proof. Condition (i) is easily proved, in fact we have

$$||Ax|| \geqslant ||x|| - 1 + \bar{g}(||x||)$$

and

$$||Ax|| \geqslant ||x - Tx|| - \bar{g}(||x||) \geqslant d(||x||) - \bar{g}(||x||)$$

and thus

$$\begin{aligned} ||Ax|| &\geqslant \max[||x|| - 1 + \bar{g}(||x||), d(||x||) - \bar{g}(||x||)] \\ &\geqslant \min_{0 \leqslant t \leqslant 1} \max[t - 1 + \bar{g}(t), d(t) - \bar{g}(t)]. \end{aligned}$$

To prove (ii) first note that, since $\gamma < 1$, $\bar{g}'$ is a positive, increasing function. Therefore the inequality

$$|\bar{g}(a) - \bar{g}(b)| \leqslant |b - a| \bar{g}'(\max(a, b))$$

holds. The Lipschitz constant of A can be evaluated as follows:

$$Ax - Ay = (x - y) - (1 - \bar{g}(||x||)(Tx - Ty) + Ty(\bar{g}(||x||) - \bar{g}(||y||)),$$

we have

$$||Ax - Ay|| \leqslant ||x - y|| + (1 - \bar{g}(||x||)L||x - y|| + |\bar{g}(||x||) - \bar{g}(||y||)|$$

and also similarly

$$||Ax - Ay|| \leqslant ||x - y|| + (1 - \bar{g}(||y||)L||x - y|| + |\bar{g}(||x||) - \bar{g}(||y||)|.$$

These two inequalities imply that

$$\begin{aligned} ||Ax - Ay|| \leqslant\ & ||x - y|| + |\bar{g}(||x||) - \bar{g}(||y||)| \\ & + L||x - y|| \min(1 - \bar{g}(||x||), 1 - \bar{g}(||y||)) \\ =\ & ||x - y|| + |\bar{g}(||x||) - \bar{g}(||y||)| + L||x - y||(1 - \max(\bar{g}(||x||), \bar{g}(||y||)) \\ =\ & ||x - y|| + |\bar{g}(||x||) - \bar{g}(||y||)| + L||x - y||(1 - \bar{g}(\max(||x||, ||y||))) \\ \leqslant\ & ||x - y|| + |||x|| - ||y|||\bar{g}'(\max(||x||, ||y||)) \\ & + L||x - y||(1 - \bar{g}(\max(||x||, ||y||))) \\ \leqslant\ & ||x - y|| \max_{0 \leqslant t \leqslant 1} [1 + \bar{g}'(t) + L - L\bar{g}(t)] = \left(1 + \frac{L(1 - \gamma)}{1 - e^{-L}}\right) ||x - y||. \end{aligned}$$

Finally for (iii) we have:

$$\begin{aligned} ||Rx - Ry|| &= \left|\left| \frac{Ax}{||Ax||} - \frac{Ay}{||Ay||} \right|\right| = \left|\left| P_R\left(\frac{Ax}{M}\right) - P_R\left(\frac{Ay}{M}\right) \right|\right| \leqslant \frac{h(X)}{M} ||Ax - Ay|| \\ &\leqslant \frac{h(X)}{M} \left(1 + \frac{L(1 - \gamma)}{1 - e^{-L}}\right) ||x - y||, \end{aligned}$$

P_R and $h(X)$ being, respectively, the radial projection and constant. □

4. The case $\psi_X(k) = 1 - 1/k$

Once the optimal choice of the function $\bar{g}$ is made, one has to work to obtain a good estimate of the constant M with a careful choice of the map T and of the function d. Here is a simple way: let $T: B(X) \to B(X)$ be a map in $\mathscr{L}(k)$ such that $\|x - Tx\| \geqslant \psi_X(k) - \varepsilon$ for any $x \in B(X)$. By the arbitrariness of ε, we can choose

$$M = \min_{0 \leqslant t \leqslant 1} \max[t - 1 + \bar{g}(t), \psi_X(k) - \bar{g}(t)].$$

Since the knowledge of the function ψ_X is "very poor" especially for lower bounds, this estimate is not very useful with the only exception of the extreme case, that is when $\psi_X(k) = 1 - 1/k$. It is known that, for example, the spaces c_0 and $C[a, b]$ have this property.

Assume now that $\gamma < 1 - \frac{1}{2k}$; if $\psi_X(k) = 1 - 1/k$ we obtain:

$$M = \min_{0 \leqslant t \leqslant 1} \max[t - 1 + \bar{g}(t), 1 - 1/k - \bar{g}(t)] = \frac{\delta}{2} - \frac{1}{2k},$$

where $\delta = \delta(k, \gamma)$ is the unique solution in $(0, 1)$ of the equation:

$$t - 1 + \bar{g}(t) = 1 - 1/k - \bar{g}(t).$$

Theorem 2. *Suppose that X is a space such that $\psi_X(k) = 1 - 1/k$ and let R be as in Proposition 2. Then R is a Lipschitz retraction of $B(X)$ onto $S(X)$ in $\mathscr{L}(Q)$, where*

$$Q = \frac{2kh(X)}{k\delta - 1}\left(1 + \frac{k(1 - \gamma)}{1 - e^{-k}}\right).$$

Noting that $h(X) \leqslant 2$, we can evaluate the expression of Q. A numerical minimization on the parameters k and γ gives

$$k_0(X) < 30.84$$

$(30.8322$ for $k = 5.81, \gamma = 0.46)$.

This result improves the result in [B-G] for the general case $\psi_X(k) = 1 - 1/k$ $(k_0(X) \leqslant 37.74)$ and the result in [B3] where it is proved that $k_0(c_0) \leqslant 35.18$; however it does not improve the result in [G2] where it is proved that $k_0(C[0, 1]) \leqslant 23.31$.

Remark. We do not know if $\psi_X(k) = 1 - 1/k$ implies $h(X) = 2$. We recall that $h(l_1) = 2$ and $\psi_{l_1}(k) < 1 - 1/k$ (see [B1]).

5. The case of l_1

As we have recalled in the last remark the space l_1 does not fit in our Theorem 1. In fact the function ψ is unknown for this space. In [B1] it is shown that

$$\psi_{l_1}(k) \geqslant \begin{cases} (3-2\sqrt{2})(k-1) & \text{if } 1 \leqslant k \leqslant 2+\sqrt{2}, \\ 1-2/k & \text{if } 2+\sqrt{2} < k \end{cases}$$

(for an upper bound see [G-K]).

In [B1], using the map defined below, it is proved that $k_0(l_1) < 31.64$.

Using the same map we improve this estimation.

Let us define $T_1 : B(l_1) \to B(l_1)$ by

$$T_1 x = T_1(x_1, x_2, x_3, \dots) = (|x_1|, |x_2|, |x_3|, \dots)$$

and $T_2 : T_1(B(l_1)) \to S^+(l_1)$ (here $S^+(l_1) = \{x \in S(l_1) \colon x_i \geqslant 0\}$) by

$$T_2 x = T_2(x_1, x_2, x_3, \dots) = (|x_1| + 1 - ||x||, |x_2|, |x_3|, \dots);$$

assuming that $k > 1$, for any $x \in S^+(l_1)$ there exists a maximal index $i_0(x)$ for which

$$\sum_{j=i_0(x)}^{\infty} x_j > 1/k$$

and there exists a unique $\mu(x) \in [0, 1)$ such that

$$\mu(x) x_{i_0(x)} + \sum_{j=i_0(x)+1}^{\infty} x_j = 1/k.$$

The map $T_3 : S^+(l_1) \to S^+(l_1)$ is then defined by

$$T_3 x = T_3(x_1, x_2, x_3, \dots) = k(0, \dots, 0, \mu(x) x_{i_0(x)}, x_{i_0(x)+1}, \dots),$$

where 0 appears $i_0(x)$ times.

Proposition 3. *Let $T = T_3 \circ T_2 \circ T_1$, then $T \in \mathscr{L}(2k)$ and for $x \in B(l_1)$ we have:*

$$||Tx - x|| \geqslant 1 + ||x|| - 2/k.$$

Proof. The proof that $T \in \mathscr{L}(2k)$ is in [G-K]. Let $x \in B$ and set $x' = T_2T_1x$ then

$$\begin{aligned}
||Tx - x|| &= \sum_{j=1}^{i_0(x')} |x_j| + |k\mu(x)|x_{i_0(x')}| - x_{i_0(x')+1}| + \sum_{j=i_0(x')+1}^{\infty} |k|x_j| - x_{j+1}| \\
&\geqslant \sum_{j=1}^{i_0(x')} |x_j| + k\mu(x)|x_{i_0(x')}| - |x_{i_0(x')+1}| \\
&\quad + k \sum_{j=i_0(x')+1}^{\infty} |x_j| - \sum_{j=i_0(x')+1}^{\infty} |x_{j+1}| \\
&= ||x|| - \sum_{j=i_0(x')+1}^{\infty} |x_j| + 1 - \sum_{j=i_0(x')+1}^{\infty} |x_j| = 1 + ||x|| - 2 \sum_{j=i_0(x')+1}^{\infty} |x_j| \\
&\geqslant 1 + ||x|| - 2/k. \qquad \square
\end{aligned}$$

Since for $x \in B(l_1)$ we have $||Tx|| = 1$ we obtain

$$||Tx - x|| \geqslant ||Tx|| - ||x|| = 1 - ||x||$$

and so

$$||Tx - x|| \geqslant \max(1 - ||x||, 1 + ||x|| - 2/k).$$

Thus for the function d in Proposition 2 we can use the function: $d(t) = \max(1 - t, 1 + t - 2/k)$ and so we obtain:

$$M = M(k, \gamma) = \min_{0 \leqslant t \leqslant 1} \max[t - 1 + \bar{g}(t), 1 - t - \bar{g}(t), 1 + t - 2/k - \bar{g}(t)].$$

Replacing in formula (1) M with the above expression and $h(X)$ with 2 we obtain a retraction with Lipschitz constant:

$$\frac{2}{M(k,\gamma)}\left(1 + \frac{2k(1-\gamma)}{1 - e^{-2k}}\right).$$

A numerical minimization on the parameters k and γ gives

$$k_0(l_1) < 22.45$$

(22.44850 for $k = 2.857$, $\gamma = 0.061$).

6. The case of Hilbert space

As in the case of l_1, in the Hilbert space $L^2[0,1]$ we will work with a particular map T_1 (see [B2]). Let $f \in L^2 = L^2[0,1]$, $k \geqslant 1$; $T_1 : L^2 \to S(L^2)$ is defined by

$$(T_1 f)(t) = \begin{cases} 1 + k|f(t)| & \text{if } 0 \leqslant t \leqslant t(f), \\ 0 & \text{if } t(f) < t \leqslant 1, \end{cases}$$

where $t(f)$ is the unique solution in $[0,1]$ of the equation:

$$\int_0^t (1 + k|f(s)|)^2 \, ds = 1.$$

Proposition 4. *The map T_1 has the following properties*:

(i) *For every $f, g \in L^2$,*

$$\|T_1 f - T_1 g\|^2 \leqslant 2k\|f - g\|.$$

(ii) *For every $f \in L^2$ and every real number α,*

$$\|f - \alpha T_1 f\|^2 \geqslant \alpha^2 + \|f\|^2 - \frac{2|\alpha|}{k}.$$

Proof. To prove (i) let $f, g \in L^2$ and suppose $t(f) \leqslant t(g)$. Then

$$\begin{aligned}
\|T_1 f - T_1 g\|^2 &= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_{t(f)}^{t(g)} (1 + k|g|)^2 \\
&= \int_0^{t(f)} (k|f| - k|g|)^2 + 1 - \int_0^{t(f)} (1 + k|g|)^2 \\
&= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_0^{t(f)} (1 + k|f|)^2 - \int_0^{t(f)} (1 + k|g|)^2 \\
&= \int_0^{t(f)} (2k^2|f|^2 - 2k^2|f||g| + 2k(|f| - |g|)) \\
&= \int_0^{t(f)} 2k(|f| - |g|)(1 + k|f|) \leqslant 2k \int_0^{t(f)} |f - g|(1 + k|f|) \\
&\leqslant 2k\left(\int_0^{t(f)} |f - g|^2\right)^{1/2}\left(\int_0^{t(f)} (1 + k|f|)^2\right)^{1/2} \leqslant 2k\|f - g\|.
\end{aligned}$$

(ii) Let $f \in L^2$ then

$$\begin{aligned} \|f - \alpha T_1 f\|^2 &= \int_0^{t(f)} (f - \alpha - \alpha k|f|)^2 + \int_{t(f)}^1 f^2 \\ &= \int_0^{t(f)} (f^2 + \alpha^2 + \alpha^2 k^2 f^2 - 2\alpha f - 2\alpha k f|f| + 2\alpha^2 k|f|) \\ &\quad + \|f\|^2 - \int_0^{t(f)} f^2 \\ &= \|f\|^2 + \int_0^{t(f)} \alpha^2(1 + k^2 f^2 + 2k|f|) - \int_0^{t(f)} \alpha(2f + 2kf|f|) \\ &\geqslant \|f\|^2 + \alpha^2 \int_0^{t(f)} (1 + k|f|)^2 - |\alpha| \int_0^{t(f)} (2|f| + 2kf^2) \\ &\geqslant \|f\|^2 + \alpha^2 - |\alpha| \int_0^{t(f)} (4|f| + 2kf^2) \geqslant \|f\|^2 + \alpha^2 - \frac{2|\alpha|}{k}. \end{aligned}$$

The last inequality is true because

$$1 = t(f) + 2k \int_0^{t(f)} |f| + k^2 \int_0^{t(f)} |f|^2$$

from which

$$2/k \geqslant 2\frac{1 - t(f)}{k} = 4 \int_0^{t(f)} |f| + 2k \int_0^{t(f)} |f|^2. \qquad \square$$

Remark. In [B2] it is proved that $\|T_1 f - T_1 g\|^2 \leqslant k^2\|f - g\|^2 + 2k(k+1)\|f - g\|$ and that $\|T_1 f - f\|^2 \geqslant (1 - 1/k)^2$.

The above described map is not a Lipschitzian map. In order to proceed we will use a technique introduced in [K-W]: first we restrict the map to a special subset $\tilde{W}$ of $B(L^2[0,1])$ where the map is Lipschitzian, then we extend this restriction to the whole space using the Kirzbraun extension theorem. We will make a better choice of the subset $\tilde{W}$ than in [K-W] (see also the same method in [B2,G-K]). In fact we will use the following theorem:

Theorem 3 (Klee [Kl]). *Let ξ be an infinite cardinal number for which $\xi^{\aleph_0} = \xi$. Then $l^2(\xi)$ contains a $\sqrt{2}$-dispersed proximinal set $\tilde{W}$ such that* $\inf\{\|x - w\|: w \in \tilde{W}\} \leqslant 1$ *for all $x \in l^2(\xi)$.*

Choose $\varepsilon > 0$ and consider in $l^2(\xi)$ the set $W = \varepsilon\tilde{W}$. Obviously if $x, y \in W$ we have:

$$\|x - y\| > \varepsilon\sqrt{2}$$

and for every x there exists a $z \in W$ such that

$$||x - z|| \leqslant \varepsilon.$$

We embed $L^2[0,1]$ in $l^2(\xi)$ as a closed subspace and we denote by P the orthogonal projection onto it. If $T_2 = T_1 P$ then T_2 has properties (i) and (ii) of Proposition 4. Indeed we have

$$||T_2 x - T_2 y||^2 \leqslant 2k||Px - Py|| \leqslant 2k||x - y||$$

and

$$||x - \alpha T_2 x||^2 = ||x - Px + Px - \alpha T_1 Px||^2 = ||x - Px||^2 + ||Px - \alpha T_1 Px||^2$$

$$\geqslant ||x - Px||^2 + \alpha^2 + ||Px||^2 - \frac{2|\alpha|}{k} = \alpha^2 + ||x||^2 - \frac{2|\alpha|}{k}.$$

Call now T_3 the restriction of T_2 to W. Then $T_3 \in \mathscr{L}(\sqrt{\frac{\sqrt{2}k}{\varepsilon}})$. Indeed if $x, y \in W$ we have

$$||T_3 x - T_3 y|| \leqslant \sqrt{2k||x - y||} < \sqrt{\frac{2k||x - y||^2}{\sqrt{2}\varepsilon}} = \sqrt{\frac{\sqrt{2}k}{\varepsilon}}||x - y||.$$

Using the Kirzbraum theorem (see [Ki]) we extend T_3 to all $l^2(\xi)$ keeping the same Lipschitz constant, we shall call this extension T_4. Notice that T_4 takes values in $\overline{co}(T_3(B(l^2(\gamma)))) \subset \overline{co}(S(L^2[0,1])) = B(L^2[0,1])$. Finally denote by T the restriction of T_4 to L^2. Obviously, T is a map from $L^2[0,1]$ to $B(L^2[0,1])$ belonging to $\mathscr{L}(\sqrt{\frac{\sqrt{2}k}{\varepsilon}})$. Now we take $x \in B(L^2)$. We evaluate directly

$$Ax = x - (1 - \bar{g}(||x||))Tx.$$

Choose $z \in W$ such that $||x - z|| \leqslant \varepsilon$; note that $z \in W$ implies $||T_4 z|| = 1$. So we have:

$$||Ax|| = ||x - (1 - \bar{g}(||x||))Tx|| = ||x - (1 - \bar{g}(||x||))(T_4 x - T_4 z + T_4 z)||$$

$$\geqslant (1 - \bar{g}(||x||))||T_4 z|| - ||x|| - (1 - \bar{g}(||x||))||T_4 x - T_4 z||$$

$$\geqslant (1 - \bar{g}(||x||)) - ||x|| - (1 - \bar{g}(||x||))\sqrt{\frac{\sqrt{2}k}{\varepsilon}}||x - z||$$

$$\geqslant (1 - \bar{g}(||x||))(1 - \sqrt{\sqrt{2}k\varepsilon}) - ||x||.$$

We also have

$$\begin{aligned}||Ax|| &= ||x-(1-\bar{g}(||x||))T_4x||\\ &= ||z-(z-x)-(1-\bar{g}(||x||))T_4z-(1-\bar{g}(||x||))(T_4x-T_4z)||\\ &\geqslant ||z-(1-\bar{g}(||x||))T_4z||-||x-z||-(1-\bar{g}(||x||))||T_4x-T_4z||\\ &\geqslant \sqrt{||z||^2+(1-\bar{g}(||x||))^2-\frac{2(1-\bar{g}(||x||))}{k}}-\varepsilon-(1-\bar{g}(||x||))\sqrt{\sqrt{2}k\varepsilon}\end{aligned}$$

and since

$$||z|| \geqslant ||x||-||z-x|| \geqslant ||x||-\varepsilon,$$

assuming $||x||>\varepsilon$ we obtain

$$\begin{aligned}||Ax|| \geqslant & \sqrt{(||x||-\varepsilon)^2+(1-\bar{g}(||x||))^2-\frac{2(1-\bar{g}(||x||))}{k}}\\ & -\ \varepsilon-(1-\bar{g}(||x||))\sqrt{\sqrt{2}k\varepsilon}.\end{aligned}$$

The preceding inequality gives

$$||Ax|| \geqslant M(k,\gamma,\varepsilon) = \min_{t\in[0,1]} \max(d_1(t),d_2(t)),$$

where

$$d_1(t) = (1-\bar{g}(t))(1-\sqrt{\sqrt{2}k\varepsilon})-t$$

and

$$d_2(t) = \begin{cases} d_1(t) & \text{if } 0\leqslant t\leqslant\varepsilon,\\ \sqrt{(t-\varepsilon)^2+(1-\bar{g}(t))^2-\frac{2(1-\bar{g}(t))}{k}} & \\ \quad -\varepsilon-(1-\bar{g}(t))\sqrt{\sqrt{2}k\varepsilon} & \text{if } \varepsilon<t\leqslant 1.\end{cases}$$

Consider now formula (1) with $h(X) = 1$, $L = \sqrt{\frac{\sqrt{2}k}{\varepsilon}}M(k, \gamma, \varepsilon)$, choosing the values: $k = 2.25$, $\varepsilon = 0.029$, $\gamma = -0.53$, we obtain the estimate:

$$k_0(L^2[0, 1]) < 28.99.$$

References

[B-S] Y. Benyamini, Y. Sternfeld, Spheres in infinite-dimensional normed spaces are Lipschitz contractible, Proc. Amer. Math. Soc. 88 (1983) 439–445.

[B1] K. Bolibok, Minimal dispacement and retraction problem in the space l_1, Nonlinear Anal. Forum 3 (1998) 13–23.

[B2] K. Bolibok, Construction of Lipschitzian mappings with non zero minimal displacement in spaces $L^1[0, 1]$ and $L^2[0, 1]$, Ann. Univ. Marie Curie-Sklodowska Sect. A 50 (1996) 25–31.

[B3] K. Bolibok, Construction of Lipschitzian retraction in the space c_0, Ann. Univ. Marie Curie-Sklodowska Sect. A 51 (1997) 43–46.

[B-G] K. Bolibok, K. Goebel, Note on minimal displacement and retraction problems, J. Math. Anal. Appl. 206 (1997) 308–314.

[F] C. Franchetti, The norm of the minimal projection onto hyperplanes in $L^p[0, 1]$ and the radial constant, Boll. Un. Mat. It. 4 (7) (1990) 803–821.

[G1] K. Goebel, On minimal displacement of points under lipschitzian mappings, Pacific J. Math. 48 (1973) 151–163.

[G2] K. Goebel, A way to retract balls onto spheres, J. Nonlinear Convex Anal. 2 (2001) 47–51.

[G-K] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[Ki] M.D. Kirzbraun, Über die Zussammenziehende und Lipschistsche Transformationen, Fund. Math. 22 (1934) 77–108.

[Kl] V. Klee, Do infinite-dimensional Banach spaces admit nice tilings?, Studia Scient. Math. Hungar. 21 (1986) 415–427.

[K-W] T. Komorowski, J. Wośko, A remark on the retracting of ball onto a sphere in an infinite-dimensional Hilbert space, Math. Scand. 67 (1990) 223–226.